

SMOOTHLY DEFORMED LIGHT

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Abstract

A single mode cavity is deformed smoothly to change its electromagnetic eigenfrequency. The system is modelled as a simple harmonic oscillator with varying period. The Wigner function of the problem is obtained exactly starting with a squeezed initial state. The result is evaluated for a linear change of the cavity length. The approach to the adiabatic limit is investigated. The maximum squeezing is found to occur for smooth change lasting only a fraction of the oscillational period. However, only a factor of two improvement over the adiabatic result proves to be possible. The sudden limit cannot be investigated meaningfully within the model.

1 Introduction

If the length of an electromagnetic cavity is changed, there are two meanings to the concept of adiabaticity. Firstly, the movement may be so slow that the cavity eigenfrequency varies only little during one oscillational period; this is the adiabatic limit proper. However, the process of establishing the correct oscillational frequency requires that the radiation has time for many round trips in the cavity. The cavity deformation may enter another regime, the eigenfrequency does not change appreciably over a few round trips, but it may change significantly over a single oscillational period. In this limit, we still expect the cavity mode to be described by a simple harmonic oscillator, but its frequency changes smoothly with time. If the movement is rapid compared with the cavity round trip time, the complete Maxwell equations need to be used in the calculation. Solving an eigenvalue problem with a moving boundary is a tricky problem; I do not want to discuss this situation here

The theory of a harmonic oscillator with variable frequency is a paradigmatic problem in physics. Classically it appears as a case of parametric driving, and quantum mechanically it is connected to the history of adiabatic invariants. A classical discussion is found in van Kampen [1] and of the many quantum treatments I wish to mention only Dykhne[2], Popov and Perelomov [3] and Man'ko and his collaborators [4]-[5]. Because the Heisenberg equations of motion agree with the classical ones, the quantum solution can be reduced completely to solving the classical problem; this was recently shown in an elegant way by Lo [6]. The same conclusion was formulated for the Wigner function by the present author [7] albeit in a different physical context. Squeezing introduced by time evolution has been discussed for other physical situations in Refs. [8]-[10].

2 The general problem and its solution

In a cavity of length L we assume the Hamiltonian for one radiation mode to be of the form

$$H = \frac{1}{2} (p^2 + \Omega^2(t)q^2) \quad (1)$$

where the time dependent frequency is given by

$$\Omega^2(t) = \Omega_0^2 f(t); \quad \Omega_0 = \frac{c\pi}{L_0} \quad (2)$$

L_0 is the initial length of the cavity. If we introduce the scaled variables

$$\tau = \Omega_0 t; \quad \pi = \frac{p}{\Omega_0}, \quad (3)$$

we find the Heisenberg equations of motion using the canonical commutation relations between p and q

$$\dot{q} = \pi; \quad \dot{\pi} = -f(\tau)q, \quad (4)$$

where the dot denotes derivation with respect to τ . Integrating these equations gives the solution for the Heisenberg variables as has been discussed in the literature.

In the Schrödinger picture we obtain the equation of motion for the Wigner function in the form

$$\frac{\partial W}{\partial \tau} + \pi \frac{\partial W}{\partial q} - f(\tau)q \frac{\partial W}{\partial \pi} = 0. \quad (5)$$

Its characteristics are the very Eqs. (4), but now they are classical relations between c-numbers. In order to solve (5) we proceed as in Ref. [7] and define the fundamental system of solutions w_1 and w_2 such that

$$w_1(0) = \dot{w}_2(0) = 1 \quad (6)$$

$$\dot{w}_1(0) = w_2(0) = 0.$$

Their Wronskian is a constant of the motion equalling unity. We assume the mode in the cavity to initially be in the squeezed state having the Wigner function

$$W_0(q_0, \pi_0) = C \exp \left[-\frac{(q_0 - \bar{q})^2}{s^2 b^2} - \frac{s^2}{b^2} (\pi_0 - \bar{\pi})^2 \right]. \quad (7)$$

Expressing the general solution of (4) in terms of the solutions (6)

$$q = q_0 w_1(\tau) + \pi_0 w_2(\tau) \quad (8)$$

$$\pi = \dot{q} = q_0 \dot{w}_1(\tau) + \pi_0 \dot{w}_2(\tau)$$

and inserting q_0 and π_0 from (8) into Eq. (7) we obtain the required solution of (5)

$$W(q, \pi, \tau) = C \exp \left[-\frac{(\dot{w}_1 q - w_2 \pi - \bar{q})^2}{s^2 b^2} \right] \times \exp \left[-\frac{s^2}{b^2} (w_1 \pi - \dot{w}_1 q - \bar{\pi})^2 \right]. \quad (9)$$

Calculating the marginal distribution for the variable q we obtain

$$\begin{aligned} \bar{W}_\pi(q, \tau) &= \int d\pi W(q, \pi, \tau) \\ &= C \exp \left[-\frac{(q - q(\tau))^2}{b^2 \sigma^2(\tau)} \right]. \end{aligned} \quad (10)$$

The Wigner function thus progresses along the classical trajectory according to

$$q(\tau) = w_1(\tau)\bar{q} + w_2(\tau)\bar{\pi} \quad (11)$$

and its spreading is given by

$$\sigma^2(\tau) = w_1^2(\tau)s^2 + \frac{w_2^2(\tau)}{s^2}. \quad (12)$$

At the initial time the squeezing is given by s^2 , but at the final time, after the change of the cavity length, the result is determined by the values of w_1 and w_2 at the end of the interaction. It is generally agreed, that in the adiabatic limit proper, the change of the squeezing must be small, see e.g. Graham [11]. In the next Section we will investigate a simple model, where we can see how the situation is changed if the motion is smooth, but not necessarily adiabatic with respect to the oscillational frequency.

3 Linear change of cavity length

We now assume that the length of the cavity is changed linearly, viz

$$L(t) = L_0 + \lambda t = L_0 + \lambda \tau / \Omega_0. \quad (13)$$

The characteristic time scale of the cavity change is given by

$$|t_0| = \frac{L_0}{|\lambda|} = \frac{\Omega_0}{|\dot{\Omega}(t)|} \quad (14)$$

which goes to infinity for properly adiabatic motion. Negative λ means that the cavity is made to contract.

With these definitions the function $f(t)$ becomes

$$f(t) = \frac{L_0^2}{(L_0 + \lambda t)^2} = \frac{1}{(1 + (t/t_0))^2}. \quad (15)$$

Relations (4) give the equation

$$\bar{q} + f(\tau)q = 0, \quad (16)$$

which has to be solved with the initial conditions (6). For the given function (15), this becomes a Fuchsian problem with two singularities and the solution can be obtained in a straightforward way.

We introduce the variables

$$A = \sqrt{\Omega_0^2 L_0^2 / \lambda^2 - \frac{1}{4}} \quad (17)$$

$$T = \frac{t}{t_0} = \frac{\tau}{\Omega_0 t_0} = \frac{\lambda t}{L_0} = \frac{L(t) - L_0}{L_0}.$$

With these definitions the fundamental solutions (6) are given by the expressions

$$w_1(\tau) = \sqrt{1+T} \left\{ \cos [A \log(1+T)] - \frac{1}{2A} \sin [A \log(1+T)] \right\} \quad (18)$$

$$w_2(\tau) = \frac{\Omega_0 t_0}{A} \sqrt{1+T} \sin [A \log(1+T)].$$

Regarding T as a function of τ , we can easily see that these functions constitute a solution to the problem. Exciting the cavity state by a classical source, we will find it in a coherent state with $s = 1$ in (7). The width as a function of time becomes

$$\sigma^2(t) = w_1^2(\tau) + w_2^2(\tau). \quad (19)$$

Before we proceed to consider the consequences of the exact expression (18) for the width (19), we look at the adiabatic limit proper, i.e. $\lambda \Rightarrow 0$. Then we find

$$A \log(1+T) \Rightarrow \frac{\Omega_0 L_0}{\lambda} \frac{\lambda t}{L_0} = \Omega_0 t \quad (20)$$

$$\frac{\Omega_0 t_0}{A} \Rightarrow 1.$$

With these results, the equations (18) go over into

$$w_1(\tau) = \sqrt{1+T} \cos \Omega_0 t \quad (21)$$

$$w_2(\tau) = \sqrt{1+T} \sin \Omega_0 t.$$

Remembering that Eq. (17) implies

$$\sqrt{1+T} = \sqrt{\frac{L(t)}{L_0}} = \sqrt{\Omega_0 / \Omega(t)} \quad (22)$$

we find that the results (21) follow from a simple application of the WKB-method to the equation (16). Inserting these results into the width (19) we find

$$\sigma^2(t) = \frac{\Omega_0}{\Omega(t)}. \quad (23)$$

As we cannot hope to change the oscillational frequency by a large fraction, we reach the conclusion that no large amount of squeezing can be achieved in the adiabatic regime proper. This agrees with conclusions arrived at in earlier treatments, in particular the adiabatic invariance of $\Omega\sigma^2$ has been found, see e.g. Ref. [11].

Another peculiarity of the result (23) is that no trace of the oscillational behaviour survives. If the parameter A is not too large, the situation changes. Because of the second term in w_1 of Eq. (18), oscillations appear in the width. To see how much squeezing they can achieve, we write the solutions (18) in the form

$$\begin{aligned} w_1 &= \sqrt{1+T} \left[\cos \varphi - \frac{1}{2A} \sin \varphi \right] \\ w_2 &= \frac{\Omega_0 t_0}{A} \sqrt{1+T} \sin \varphi. \end{aligned} \quad (24)$$

Here φ is the argument of the trigonometric functions in Eqs. (18). The width (19) then becomes

$$\sigma^2(t) = \frac{\Omega_0}{\Omega(t)} \left[1 - \frac{1}{2A} \sin 2\varphi + \frac{1}{2A^2} \sin^2 \varphi \right]. \quad (25)$$

For $A \Rightarrow \infty$ this reproduces (23). The expression has a minimum for each fixed value of the parameter A , but for large A , this approaches the adiabatic limiting value (23). For example $A = 1$ gives the minimum value 0.69 for the expression in square brackets in (25). This occurs at the time when $\varphi = 0.55$.

The best possible values for the squeezing are obtained with a very small A , in which case the minimum occurs for early times, $\varphi \approx 0$. The expression (25) can then be written

$$\sigma^2(t) = \frac{\Omega_0}{2\Omega(t)} \left[1 + \left(\frac{\varphi}{A} - 1 \right)^2 \right] \geq \frac{\Omega_0}{2\Omega(t)}. \quad (26)$$

which is not a large improvement over (23). The minimum also occurs for a small parameter A , in which case we rapidly approach the breakdown of the validity of the theory. For very small A , the expression (17) gives

$$\Omega_0 t_0 \simeq \frac{1}{2}, \quad (27)$$

which is not in the adiabatic regime proper. The minimum then occurs at times when

$$\varphi \approx \Omega_0 t \approx A < \Omega_0 t_0 \approx \frac{1}{2}. \quad (28)$$

Thus we have to change the cavity eigenfrequency in a time less than the oscillation period. This cannot obviously be achieved by mechanical means, and even using some electronic switching to change the effective path length through the cavity, we can attempted this only in the microwave region. However, as the advantage of the method is expected to be small, there seems to be little motivation to solve the technical problems involved.

4 Discussion

We have solved the problem of the deformation of an intracavity field during a smooth change of the cavity eigenfrequency. Even if we are allowed to depart from the strict adiabatic limit, the expected squeezing remains modest. The calculation cannot be taken to the sudden limit, because then the simple harmonic oscillator description is no longer valid. The complete Maxwell equations must be treated in that case. In this aspect our problem differs from the corresponding Schrödinger equation [12]-[14] where both the sudden and the adiabatic limit can be handled in the same way.

5 Acknowledgements

I want to thank Professors Man'ko and Trifonov for pointing out some computational errors in an early version of this work.

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